

Hermitian quark mass matrices with four texture zeros

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Abstract

We provide a complete and systematic analysis of hermitian, hierarchical quark mass matrices with four texture zeros. Using triangular mass matrices, each pattern of texture zeros is readily shown to lead to a definite relation between the CKM parameters and the quark masses. Nineteen pairs are found to be consistent with present data, and one other is marginally acceptable. In particular, no parallel structure between the up and down mass matrices is found to be favorable with data.

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I. INTRODUCTION

The standard model is known to have a large number of parameters. However, most of them are contained in the pair of quark mass matrices, M_U and M_D . Of the thirty-six parameters in these two complex matrices, only ten [six quark masses, three Cabibbo-Kobayashi-Maskawa (CKM) angles and a CP phase] are physical. This redundancy arises from unphysical, right-handed (RH) rotations in both the U - and D -sectors, in addition to common left-handed (LH) rotations, which cancel out in the CKM matrix, $V^{\text{CKM}} = V_U^\dagger V_D$, where $V_U^\dagger M_U U_U = M_U^{\text{diag}}$ and $V_D^\dagger M_D U_D = M_D^{\text{diag}}$. To eliminate these unphysical degrees of freedom, it was pointed out earlier [1] that RH rotations can be used to reduce both M_U and M_D to the upper triangular form, which in the hierarchical basis exhibits most clearly the quark masses and left-handed (LH) rotation angles,

$$M_U U_U^R = M_U^t = \begin{pmatrix} X & X & X \\ 0 & X & X \\ 0 & 0 & X \end{pmatrix}, \quad (1)$$

$$M_D U_D^R = M_D^t = \begin{pmatrix} X & X & X \\ 0 & X & X \\ 0 & 0 & X \end{pmatrix}. \quad (2)$$

The matrices U_U^R and U_D^R can be constructed explicitly as $R_{13}R_{23}R_{12}$, such that R_{13} eliminates the (3,1) element of M , etc. Another way to obtain the triangular matrix elements directly is by a geometric argument. Let us write a general (real) matrix M in the form:

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}. \quad (3)$$

We may regard the three rows as components of the vectors \vec{a} , \vec{b} and \vec{c} . Then the transformation into the triangular form amounts to a rotation into a new coordinate system where the

axes are aligned in the directions of the unit vectors: $(\vec{b} \times \vec{c})/|\vec{b} \times \vec{c}|$, $\vec{c} \times (\vec{b} \times \vec{c})/(|\vec{c}| \cdot |\vec{b} \times \vec{c}|)$, and $\vec{c}/|\vec{c}|$, respectively. It follows that

$$M^t = \begin{pmatrix} \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|} & \frac{(\vec{a} \times \vec{c}) \cdot (\vec{b} \times \vec{c})}{|\vec{c}| \cdot |\vec{b} \times \vec{c}|} & \frac{\vec{a} \cdot \vec{c}}{|\vec{c}|} \\ 0 & \frac{|\vec{b} \times \vec{c}|}{|\vec{c}|} & \frac{\vec{b} \cdot \vec{c}}{|\vec{c}|} \\ 0 & 0 & |\vec{c}| \end{pmatrix}. \quad (4)$$

For complex vectors \vec{a}, \vec{b} and \vec{c} , each transforms as a $\mathbf{3}$ under RH SU(3) rotations. From $[\mathbf{3} \times \mathbf{3}]_{\text{anti}} \sim \bar{\mathbf{3}}$, $[\mathbf{3} \times \mathbf{3} \times \mathbf{3}]_{\text{anti}} \sim \mathbf{1}$, etc., we readily find:

$$M^t = \begin{pmatrix} \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|} & \frac{(\vec{a} \times \vec{c}) \cdot (\vec{b}^* \times \vec{c}^*)}{|\vec{c}| \cdot |\vec{b} \times \vec{c}|} & \frac{\vec{a} \cdot \vec{c}^*}{|\vec{c}|} \\ 0 & \frac{|\vec{b} \times \vec{c}|}{|\vec{c}|} & \frac{\vec{b} \cdot \vec{c}^*}{|\vec{c}|} \\ 0 & 0 & |\vec{c}| \end{pmatrix}. \quad (5)$$

Note that this construction is unique up to a diagonal phase matrix multiplying from the right: $M^t \rightarrow M^t P$, $P = \text{diagonal phase matrix}$.

After we have transformed M_U and M_D to the upper triangular form, we may reduce them further by common LH rotations on both. Since there are three degrees of freedom in these rotations, we can use them to generate three additional zeros in the pair. Note that this result is only approximately correct. The LH rotations will generate small elements in the lower-left part of the matrices, which can be removed by even smaller RH rotations. Thus, the process actually consists of a sequence of rotations R_i , which can be schematically expressed as $\cdots R_3((\frac{m_1}{m_2})^2 V_{12}, (\frac{m_2}{m_3})^2 V_{23}, (\frac{m_1}{m_3})^2 V_{13}) R_1(V_{12}, V_{23}, V_{13}) M R_2(\frac{m_1}{m_2} V_{12}, \frac{m_2}{m_3} V_{23}, \frac{m_1}{m_3} V_{13}) \cdots$.

The above argument provides an algorithm to reduce any pair of M_U and M_D into upper triangular forms, with the minimal nine non-vanishing elements between the two matrices. As was shown earlier, the most interesting property of this reduction is that to a good approximation, all of the remaining nine non-vanishing matrix elements in M^U and M^D are physical and are simple products of a quark mass and some CKM matrix elements. This method can thus provide us with a powerful tool to assess the viability of any proposed mass matrices. It was applied to hermitian mass matrices with five texture zeros, and a unique pair is identified as most favorable to present data [2].

In this paper, we will apply the same technique to the study of hermitian mass matrices with four texture zeros, which generally entail one testable relation for each pair. The study of 4-texture-zero hermitian mass matrices has been a subject of interest recently [3-8]. Attentions have been focused on specific models or structures, which can lead to predictive values for the CKM matrix elements. However, due to the complexity involved in the analysis, no complete investigation has been performed so far. Using the triangular basis can greatly reduce the amount of work involved in analyzing the viability of 4-zero textures. With this efficient tool, we are able to provide the first complete study of all viable four-zero texture pairs that exhibit hierarchical structure. The results are presented in Tables I through IV.

II. HERMITIAN MASS MATRICES AND TEXTURE ZEROS

We start with quark mass matrices in the triangular form. Using the mass relations $m_u : m_c : m_t \sim \lambda^8 : \lambda^4 : 1$ and $m_d : m_s : m_b \sim \lambda^4 : \lambda^2 : 1$, and the CKM elements $V_{us} = \lambda$, $V_{cb} \sim \lambda^2$, $V_{ub} \sim \lambda^4$, and $V_{td} \sim \lambda^3$, the properly normalized Yukawa matrices for U and D can be put into the most general triangular form,

$$T^U = \begin{pmatrix} a_U \lambda^8 & b_U \lambda^6 & c_U \lambda^4 \\ 0 & d_U \lambda^4 & e_U \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^D = \begin{pmatrix} a_D \lambda^4 & b_D \lambda^3 & c_D \lambda^3 \\ 0 & d_D \lambda^2 & e_D \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

Here, all of the coefficients (a, b, \dots) are assumed to be of order unity or less. Without loss of generality, we also take the diagonal parameters (i.e. a_U, a_D, d_U, d_D) to be real throughout this paper. We offer the following remarks.

1. The hierarchical structures of T^U and T^D are manifest. As was shown earlier [1,2], we can diagonalize them approximately with only a LH rotation, $M_{U,D}^{\text{diag}} \simeq V_{U,D}^\dagger T^{U,D}$, with the matrix elements $(V_{U,D})_{ij} \simeq T_{ij}^{U,D} / T_{jj}^{U,D}$ ($i < j$).
2. The diagonal elements of T^U and T^D are essentially the quark masses. The CKM matrix elements are simply given by $V_{ij}^{\text{CKM}} \sim (V_D)_{ij} - (V_U)_{ij}$, ($i < j$). These simple

relations are lost when we go to other bases, including the hermitian basis. Triangular matrices thus stand out as the unique basis whose matrix elements correspond directly to quark masses and LH rotation angles. This feature makes the triangular form especially useful for analyzing the texture of quark mass matrices.

3. To avoid fine tuning in generating the CKM mixings, naturalness criteria [11] has been imposed when writing the above triangular matrices. This simply implies that, for the LH rotation angles, $|V_{U,D}|_{ij} \leq |V_{ij}^{\text{CKM}}| \times \mathcal{O}(1)$. Note that this condition can always be implemented by applying a common LH rotation on T^U and T^D .
4. We can put T^U and T^D in the minimal parameter basis (m.p.b.), with only three nonzero off-diagonal elements between them. This is achieved from Eq. (6) by common LH rotations to generate three zeros. For example, the (1,3) element of T^D can be set to zero by a common LH rotation $R_{13}(-c_D\lambda^3)$. This method produces ten pairs of triangular matrices, as listed in Table I of Ref. [2]. The elements of each pair consist of a simple product of quark mass and CKM elements. Nothing is unphysical in the m.p.b.. As a result, the viability of a given texture can be readily tested by turning it into one of the ten.

Turning now to hermitian matrices, the consequences of their texture zeros can be easily understood in terms of the triangular parameters of Eq. (6). For this purpose, one can start from Eq. (6) and, by a RH rotation, generate the corresponding hermitian form,

$$Y^U = \begin{pmatrix} \left(a_U + c_U c_U^* + \frac{b_U b_U^*}{d_U}\right) \lambda^8 & (b_U + c_U e_U^*) \lambda^6 & c_U \lambda^4 \\ (b_U^* + c_U^* e_U) \lambda^6 & (d_U + e_U e_U^*) \lambda^4 & e_U \lambda^2 \\ c_U^* \lambda^4 & e_U^* \lambda^2 & 1 \end{pmatrix} \times (1 + \mathcal{O}(\lambda^4)) , \quad (7)$$

$$Y^D = \begin{pmatrix} \left(a_D + \frac{b_D b_D^*}{d_D}\right) \lambda^4 & b_D \lambda^3 & c_D \lambda^3 \\ b_D^* \lambda^3 & d_D \lambda^2 & e_D \lambda^2 \\ c_D^* \lambda^3 & e_D^* \lambda^2 & 1 \end{pmatrix} \times (1 + \mathcal{O}(\lambda^2)) . \quad (8)$$

We immediately see that whereas off-diagonal hermitian zeros (e.g. Y_{13}^U , Y_{23}^U or Y_{ij}^D ($i \neq j$)) have a one-to-one correspondence to zeros in the triangular form, the diagonal hermitian zeros (i.e. Y_{11}^U , Y_{22}^U , and Y_{11}^D) entail definite relations between diagonal and off-diagonal triangular parameters (i.e. quark masses and LH rotation angles). Note that these simple relations between the hermitian and triangular forms are approximate, and follow from the hierarchical structure in Eq. (6).

We are thus led to the following procedure in analyzing hermitian texture zeros. First, we obtain all possible textures with certain number of zeros for Y^U and Y^D by referring to Eq. (7) and Eq. (8), as listed in Tables I and III. Then we can list all the 4-texture-zero pairs. Second, each pair is transformed into one of the ten triangular forms in the m.p.b., using common LH rotations when necessary. Possible relations between quark masses and mixing can then be read-off for each pair by referring to Table I of Ref. [2]. Finally, we arrive at all the viable textures after confronting the prediction of each pair with data. In the next section, we apply this procedure to the analysis of hermitian matrices with four texture zeros.

III. ANALYSES OF MATRICES WITH FOUR TEXTURE ZEROS

The study of viable 4-texture-zero pairs is straightforward by following the procedure outlined above. For convenience in our analysis and presentation, we categorize the mass matrix pairs (M_U, M_D) according to whether or not the $(1, 1)$ matrix element is zero:

- 1) $M_U^{11} = 0, M_D^{11} = 0$,
- 2) $M_U^{11} \neq 0, M_D^{11} = 0$,
- 3) $M_U^{11} = 0, M_D^{11} \neq 0$,
- 4) $M_U^{11} \neq 0, M_D^{11} \neq 0$.

The results are presented in Tables I-IV, which contain all the viable 4-texture-zero pairs. It is seen from the Tables that almost all of the viable textures have $M_D^{11} = 0$.

The first category involves eight types of hermitian matrices, which are listed in Table I

along with their corresponding triangular forms. These hermitian matrices contain 1, 2 and 3 texture zeros for M_U and/or M_D . Their corresponding triangular matrices can be simply obtained from Eqs. (7) and (8). They are then paired up to form possible patterns for the quark mass matrices. Note that M_2, M_3, M_6, M_7 and M_8 are the ones appearing in the analysis of five texture-zero matrices [9,10,2]. One notable feature about the pairing is that M_U and M_D allow different texture zeros, as can be seen from Eqs. (7) and (8). For example, whereas $M_D^{22} \neq 0$ (unless we give up naturalness), $M_U^{22} = 0$ is allowed due to the much larger mass hierarchy in the up-quark sector. As a second example, $M_U^{11} = M_U^{22} = 0$ is allowed but the same relation is not valid for M_D again because of the different mass hierarchies. As another tip for the pairing, a quick check on the CKM matrix element can be useful in the screening of possible candidates. For instance, the (M_4, M_4) pair is out because both their (2,3) elements are zero, which in turns gives $V_{cb} \simeq 0$. The same can not be said about hermitian pairs with vanishing (1,3) elements. Here, V_{ub} and V_{td} may or may not vanish since they are of higher order in λ , and may be induced from a combined R_{12} and R_{23} rotations. In this latter case, a further investigation is required.

The viability of each (M_U, M_D) pair can be assessed by converting them, through common LH rotations if necessary, into one of the ten triangular pairs in the m.p.b. as listed in Table I of Ref. [2]. Testable relations can then be obtained. To illustrate the method, we first study the (M_8, M_1) pair in detail. The triangular forms for M_8 and M_1 are given by:

$$M_8 \rightarrow M_U^t \simeq \begin{pmatrix} a_U \lambda^8 & b_U \lambda^6 & 0 \\ 0 & d_U \lambda^4 & e_U \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (9)$$

$$M_1 \rightarrow M_D^t \simeq \begin{pmatrix} a_D \lambda^4 & b_D \lambda^3 & c_D \lambda^3 \\ 0 & d_D \lambda^2 & e_D \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10)$$

Here, $d_U = -|e_U|^2$. This (M_U^t, M_D^t) pair has eleven nonzero elements, and two more zeros are needed to put it into one of the ten triangular pairs in the m.p.b.. There are several

ways to achieve this. For example, a common LH 2-3 rotation with $\theta_{23} \simeq -e_U \lambda^2$ will set the (2, 3) element of M_U to zero, and a common LH 1-2 rotation with $\theta_{12} \simeq -\frac{b_D}{d_D} \lambda$ will generate a second zero at the (1, 2) position of M_D . In this way, we arrive at

$$M_U' = R_{12}\left(-\frac{b_D}{d_D}\lambda\right)R_{23}(-e_U\lambda^2)M_U^t \simeq \begin{pmatrix} a_U\lambda^8 & b_U\lambda^6 - \frac{b_D}{d_D}d_U\lambda^5 & 0 \\ 0 & d_U\lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (11)$$

$$M_D' = R_{12}\left(-\frac{b_D}{d_D}\lambda\right)R_{23}(-e_U\lambda^2)M_D^t \simeq \begin{pmatrix} a_D\lambda^4 & 0 & c_D\lambda^3 - \frac{b_D}{d_D}(e_D - e_U)\lambda^3 \\ 0 & d_D\lambda^2 & (e_D - e_U)\lambda^2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

Identifying (M_U', M_D') with the 5th triangular pattern in Table I of Ref. [2] gives,

$$\lambda^2 b_U/d_U - \lambda b_D/d_D = -V_{us}/V_{cs}. \quad (13)$$

As $M_U^{11} = M_D^{11} = 0$ in the hermitian form, we have two corresponding relations in terms of triangular parameters: $a_U = -\frac{|b_U|^2}{d_U}$ and $a_D = -\frac{|b_D|^2}{d_D}$. With these two relations, Eq. (13) can be written in the well known form,

$$\left|\frac{V_{us}}{V_{cs}}\right| \simeq \left|\sqrt{\frac{m_d}{m_s}} - e^{i\delta} \sqrt{\frac{m_u}{m_c}}\right|, \quad (14)$$

where $\delta \equiv \arg[b_U d_D / d_U b_D]$. From Eqs. (11-12) and Table I of Ref. [2], we note that the standard model CP -violation depends on additional phases besides δ , thus leaving δ a free parameter. In this sense, Eq. (14) places a constraint on the mass matrices in fixing the phase δ , but it is not a prediction in terms of physical quantities alone. We conclude that the (M_8, M_1) pair is a viable texture since Eq. (14) can be made valid with a properly chosen phase δ .

To assess the viability of any given texture, we will need to know about the quark masses and CKM elements. We use for the quark masses at m_Z values taken from Ref. [12]. The CKM matrix elements are taken from [13], except for V_{ub} and V_{td} , for which we use a recent update [14,15]:

$$\left| \frac{V_{ub}}{V_{cb}} \right|_{exp} = 0.093 \pm 0.014, \quad (15)$$

$$0.15 < \left| \frac{V_{td}}{V_{ts}} \right|_{exp} < 0.24. \quad (16)$$

Whereas Eq. (15) comes from an average of the LEP and SLD measurements, Eq. (16) comes from a standard model fit to the electroweak data. Note that Eq. (16) implies $|V_{td}| < 0.01$.

After a straightforward analysis of all possible texture pairs, we arrive at the nine viable pairs of hermitian matrices for the first category. These are listed in Table II, together with their testable relations. Note that relations R1 and R2 allow two different solutions depending on the sign of d_U , and that some textures share the same quark mass-mixing relation.

Also listed in Table II is the particular pair (pattern 10) which is parallel in structure: (M_3, M_3) . This texture pair has been the center of focus in recent studies of hermitian 4-texture-zero quark mass matrices [3–8]. To analyze the predictions of this pair, we first write down the triangular form,

$$M_3 \rightarrow M_U^t \simeq \begin{pmatrix} a_U \lambda^8 & b_U \lambda^6 & 0 \\ 0 & d_U \lambda^4 & e_U \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (17)$$

$$M_3 \rightarrow M_D^t \simeq \begin{pmatrix} a_D \lambda^4 & b_D \lambda^3 & 0 \\ 0 & d_D \lambda^2 & e_D \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

We need one more zero to put the triangular pair (M_U^t, M_D^t) in the m.p.b., and this can be attained by a common LH 2-3 rotation of $\theta_{23} \simeq e_U \lambda^2$ ($\theta_{23} \simeq e_D \lambda^2$) to set the (2, 3) element of M_U^t (M_D^t) to zero. For example, we can have

$$M_U^{t'} = R_{23}(-e_U \lambda^2) M_U^t \simeq \begin{pmatrix} a_U \lambda^8 & b_U \lambda^6 & 0 \\ 0 & d_U \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (19)$$

$$M_D^{t'} = R_{23}(-e_U \lambda^2) M_D^t \simeq \begin{pmatrix} a_D \lambda^4 & b_D \lambda^3 & 0 \\ 0 & d_D \lambda^2 & (e_D - e_U) \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (20)$$

Note that $(M_U^{t'}, M_D^{t'})$ corresponds to the 2nd of the ten triangular pairs in the m.p.b. as listed in Table I of Ref [2]. A simple comparison between them gives the following relations for the hermitian (M_3, M_3) texture:

$$\frac{V_{ub}}{V_{cb}} \simeq -\frac{b_U}{d_U} \lambda^2, \quad \frac{V_{td}^*}{V_{cb} V_{cs}} \simeq \frac{b_D}{d_D} \lambda, \quad (21)$$

and

$$\arg \left[\frac{b_U d_D}{d_U b_D} \right] \simeq \alpha \equiv \arg \left[-\frac{V_{td} V_{tb}^*}{V_{ud} V_{ub}^*} \right], \quad (22)$$

where α is the α -angle of unitarity triangle. Now the two zeros at the $(1, 1)$ position of (M_3, M_3) imply two relations among the triangular parameters: $|b_U^2| = -a_U d_U$ and $|b_D^2| = -a_D d_D$. This allows us to rewrite Eq. (21) in the form,

$$\left| \frac{V_{ub}}{V_{cb}} \right| \simeq \sqrt{\frac{m_u}{m_c}}, \quad (23)$$

$$\left| \frac{V_{td}}{V_{ts}} \right| \simeq \sqrt{\frac{m_d}{m_s}}. \quad (24)$$

Using the unitarity of the CKM matrix, the two relations in Eq. (21) can be combined to give

$$\frac{V_{us}}{V_{cs}} = \frac{V_{ub}}{V_{cb}} + \frac{V_{td}^*}{V_{cb} V_{cs}} \simeq \frac{b_D}{d_D} \lambda - \frac{b_U}{d_U} \lambda^2, \quad (25)$$

where we have assumed $\det V^{\text{CKM}} = 1$ without loss of generality. Using $|b_U^2| = -a_U d_U$, $|b_D^2| = -a_D d_D$, and Eq. (22), Eq. (25) can be cast in the form,

$$\left| \frac{V_{us}}{V_{cs}} \right| \simeq \left| \sqrt{\frac{m_d}{m_s}} - e^{i\alpha} \sqrt{\frac{m_u}{m_c}} \right|. \quad (26)$$

Note that Eq. (26) is similar but different from Eq. (14): while the phase in Eq. (26) is in principle fixed by the CP -violation parameter ϵ_K to be $\alpha \sim \pi/2$ [15,14], the phase δ in

Eq. (14) can be varied freely as the standard model CP violation depends on other phases as well. The relations in Eqs.(23), (24), and (26) were also obtained in Ref. [3–8].

Whereas Eqs. (24) and (26) are allowed by data, the prediction of Eq. (23), $V_{ub}/V_{cb} = 0.059 \pm 0.006$, is disfavored by the recent measurement (see Eq. (15)). The same conclusion has been reached for three of the five hermitian pairs in our five-texture-zero analysis [2]. It is interesting and surprising that going to four-texture-zero does not help with this problem of low V_{ub}/V_{cb} . Furthermore, even varying m_u and m_c in a reasonable range will not be able to accommodate a value of $V_{ub}/V_{cb} = 0.08$ [16]. In this regard, we are in disagreement with the analysis of Ref. [5], where much larger values ($0.083 \sim 0.099$) for V_{ub}/V_{cb} were obtained. We note also that texture zeros are not invariant under change of basis. Our result for the pair (M_3, M_3) is valid only in the hierarchical basis. Otherwise, larger values of V_{ub}/V_{cb} are allowed, as in Eqs.(23-26) of Ref. [7]. For these reasons, we exclude the (M_3, M_3) pair from the “Yes” column in Table II.

It is worthwhile to compare in detail the results of our two examples. Both the pairs (M_8, M_1) and (M_3, M_3) have four texture zeros. But the former yields one relation that is not physically predictive in nature (Eq.(14)), while the later gives rise to two independent predictions of Eqs.(23-24) (note that Eq. (26) is not independent).

In general, for a pair of hermitian matrices with four texture zeros, there are eight real parameters plus one or more phases, so we expect at most one prediction. This is the case for the pair (M_8, M_1) , where we have more than one unremovable phases and no physical prediction. The situation is different with (M_3, M_3) . From Eqs.(19-20), using $a_U = -|b_U|^2/d_U$ and $a_D = -|b_D|^2/d_D$, the independent parameters are $|b_U|$, $|b_D|$, $|d_U|$, $|d_D|$, $|e_U - e_D|$, plus two overall mass scales and one physical phase. Thus, triangularization reveals that the pair (M_3, M_3) actually contains one less parameter than expected, because only the combination $(e_U - e_D)$ enters. Put another way, we could set $e_U = 0$ (or $e_D = 0$) without any effect. This means that the physical contents of (M_3, M_3) are the same as the five texture zero pair (M_7, M_3) or (M_3, M_7) . In addition, we can see that the five-zero pair (M_8, M_3) also reduces to the same triangular form by a LH 2-3 rotation. Thus, we conclude

that all four pairs, (M_3, M_3) , (M_7, M_3) , (M_3, M_7) , and (M_8, M_3) , are physically equivalent. The analytical predictions of the last three pairs were studied in Ref. [2].

Similar analyses of the hermitian pairs with $M_U^{11} \neq 0$ and/or $M_D^{11} \neq 0$ follow directly. The results are presented in Tables III and IV. In Table III, we list the nine hierarchical hermitian matrices with $(1, 1) \neq 0$ that are used in the construction of viable hermitian pairs. The corresponding triangular matrices are also given in the table. The viable hermitian 4-texture-zero pairs constructed from Tables I and III are listed in Table IV, together with their predictions for quark mixing. Whereas there is no viable pair with both $M_U^{11} \neq 0$ and $M_D^{11} \neq 0$, nine pairs with $M_U^{11} \neq 0$ and $M_D^{11} = 0$ are found to be compatible with data. Of the hermitian matrices with $M_U^{11} = 0$ and $M_D^{11} \neq 0$, one pair is viable, and the 2nd pair, (M_2, M_{12}) , leads to the same prediction for V_{ub} as that of the 5th five-texture-zero pair (see Eq.(15) in Ref. [2]). This pair allows two different relations depending on the sign of d_U . While the plus-sign relation is marginally acceptable, the minus-sign relation is disfavored by data.

Before we leave this section, we would like to point out that some of the disfavored hermitian pairs can give rise to nontrivial predictions. For example, the four pairs, (M_1, M_{17}) and (M_i, M_{13}) ($i = 2, 4, 5$), all lead to the relation:

$$|V_{td}| \simeq \sqrt{V_{us}^2 m_c / m_t + m_u / m_t} = |V_{us}| \sqrt{m_c / m_t} \times (1 + \mathcal{O}(\lambda^2)) > 1\% . \quad (27)$$

This prediction contradicts the experimental limit of $|V_{td}| < 1\%$, as can be derived from $B\bar{B}$ mixing (see also Eq. (16)). As another example, the hermitian pair (M_2, M_{11}) entails the relation:

$$|V_{cb}| \simeq \sqrt{\frac{m_c}{m_t}} \left(1 + \frac{m_u}{m_c V_{us}^2} \right)^{-\frac{1}{2}} = \sqrt{\frac{m_c}{m_t}} \times (1 + \mathcal{O}(\lambda^2)) \sim 0.06 , \quad (28)$$

which is too large for this texture to be viable.

A notable feature about Tables I-IV is that some texture pairs lead to the same relation, like patterns 1, 2, 5, and 6. It can be shown that these texture pairs are related by weak basis transformation, and thus are physically equivalent.

IV. CONCLUSIONS

Because of the many redundancies in the quark mass matrices, a lot of work has been done in search for more restrictive patterns of matrices which are still compatible with experiments. In the literature, such considerations have focused on hermitian mass matrices with a certain number of texture zeros. It was found that the maximum allowed number of texture zeros is five. Amongst them one unique pattern has been identified which is most favorable with data. It is worthwhile to understand the situation for pairs of matrices with fewer texture zeros.

In this paper, we have investigated systematically hermitian hierarchical quark mass matrices with four texture zeros. By transforming the hermitian matrices into the triangular form in the minimal parameter basis, and using the fact that the latter contains only physical masses and CKM matrix elements, we can quickly rule out many of the possible pairs of mass matrices. For the remaining candidates, we showed that each pattern of texture zeros implies certain relations between quark masses and mixing. These relations can be used to determine the viability of each pair of mass matrices by comparing them with available data. In this way we identified nineteen pairs of mass matrices which are compatible with current data. One pair (pattern 21) is marginal, with the final verdict depending on more accurate experimental numbers. Of the nineteen viable textures, none has a parallel structure between the up and down mass matrices. In particular, the popular parallel-structure pair that has been a focus of much attention, pattern 10, leads to a low value of V_{ub}/V_{cb} , and is not favorable with present data. We also found that, by a proper LH rotation, pattern 10 can be shown to have the same physical contents as three (which are themselves equivalent) of the five pairs studied earlier [2,9]. The asymmetry between up and down quark mass matrix textures could serve as a useful guideline in search for realistic models of quark-lepton masses.

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TABLES

M_1	M_2	M_3	M_4
$\begin{pmatrix} 0 & B' & C \\ B'^* & D' & E \\ C^* & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & C \\ 0 & D' & E \\ C^* & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & B & 0 \\ B^* & D' & E \\ 0 & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & B & C \\ B^* & D & 0 \\ C^* & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} - C ^2 - \frac{ B ^2}{D} & B & C \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A' & -CE^* & C \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{ B ^2}{D} & B & 0 \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} - C ^2 - \frac{ B ^2}{D} & B & C \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$
M_5	M_6	M_7	M_8
$\begin{pmatrix} 0 & B' & C \\ B'^* & 0 & E \\ C^* & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & C \\ 0 & D & 0 \\ C^* & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & B & 0 \\ B^* & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & B & 0 \\ B^* & 0 & E \\ 0 & E^* & 1 \end{pmatrix}$
$\begin{pmatrix} \frac{ B ^2}{ E ^2} - C ^2 & B & C \\ 0 & - E ^2 & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} - C ^2 & 0 & C \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{ B ^2}{D} & B & 0 \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{ B ^2}{ E ^2} & B & 0 \\ 0 & - E ^2 & E \\ 0 & 0 & 1 \end{pmatrix}$

TABLE I. The eight hierarchical hermitian matrices with $(1,1)=0$. The corresponding triangular forms are also listed. Matrices in this table can be paired up to form quark mass matrices with 4 texture zeros. Here $A' \equiv -|C|^2(1 + \frac{|E|^2}{D})$, $B' \equiv B + CE^*$, and $D' \equiv D + |E|^2$.

Pattern	$(M_U^{11} = 0, M_D^{11} = 0)$	Relation	Viable
1	(M_1, M_7)	R1: $ V_{ub} - K \cdot V_{ts}^* ^2 \simeq \left \frac{m_u}{m_t} \pm \frac{m_c}{m_t} K^2 \right $	Yes
2	(M_2, M_3)	R1: $ V_{ub} - K \cdot V_{ts}^* ^2 \simeq \left \frac{m_u}{m_t} \pm \frac{m_c}{m_t} K^2 \right $	Yes
3	(M_2, M_4)	R2: $\left \frac{V_{us}}{V_{cs}} \right \simeq \left \sqrt{\frac{m_d}{m_s}} - e^{i\delta} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{V_{cb}^2}{(m_c/m_t) \pm V_{cb}^2}} \right $	Yes
4	(M_3, M_4)	R3: $\left \frac{V_{us}}{V_{cs}} \right \simeq \left \sqrt{\frac{m_d}{m_s}} - e^{i\delta} \sqrt{\frac{m_u}{m_c}} \right $	Yes
5	(M_4, M_3)	R1: $ V_{ub} - K \cdot V_{ts}^* ^2 \simeq \left \frac{m_u}{m_t} \pm \frac{m_c}{m_t} K^2 \right $	Yes
6	(M_5, M_3)	R1: $ V_{ub} - K \cdot V_{ts}^* ^2 \simeq \left \frac{m_u}{m_t} \pm \frac{m_c}{m_t} K^2 \right $	Yes
7	(M_6, M_1)	R4: $ V_{us} \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
8	(M_7, M_1)	R3: $\left \frac{V_{us}}{V_{cs}} \right \simeq \left \sqrt{\frac{m_d}{m_s}} - e^{i\delta} \sqrt{\frac{m_u}{m_c}} \right $	Yes
9	(M_8, M_1)	R3: $\left \frac{V_{us}}{V_{cs}} \right \simeq \left \sqrt{\frac{m_d}{m_s}} - e^{i\delta} \sqrt{\frac{m_u}{m_c}} \right $	Yes
10	(M_3, M_3)	R5: $\left \frac{V_{td}}{V_{ts}} \right \simeq \sqrt{\frac{m_d}{m_s}}$, R6: $\left \frac{V_{ub}}{V_{cb}} \right \simeq \sqrt{\frac{m_u}{m_c}}$, and R3	No

TABLE II. The ten possible candidates for hermitian, hierarchical quark mass matrices with vanishing (1,1) elements. Also listed are the testable relations derived from each pair. In writing relation R1, we have used for simplicity in presentation $V_{tb} \simeq V_{cs} \simeq V_{ud} \simeq 1$, and $K \equiv e^{i\delta} \sqrt{\frac{m_d}{m_s}} - \frac{V_{us}}{V_{cs}}$. Except for pattern 7, all other relations require properly chosen phases to be viable. The nine viable pairs lead to 4 different relations, three of which are similar: R2, R3, and R4. Relation R6 predicts a too small V_{ub}/V_{cb} , which makes pattern 10 disfavored.

M_9	M_{10}	M_{11}	M_{12}	M_{13}
$\begin{pmatrix} A'' & 0 & C \\ 0 & D' & E \\ C^* & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} A + \frac{ B ^2}{D} & B & 0 \\ B^* & D' & E \\ 0 & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} A + C ^2 & 0 & C \\ 0 & D & 0 \\ C^* & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A + \frac{ B ^2}{D} & B & 0 \\ B^* & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & 0 \\ 0 & D' & E \\ 0 & E^* & 1 \end{pmatrix}$
$\begin{pmatrix} A & -CE^* & C \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & B & 0 \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & C \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & B & 0 \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & 0 \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$
M_{14}	M_{15}	M_{16}	M_{17}	
$\begin{pmatrix} A - \frac{ B ^2}{ E ^2} & B & 0 \\ B^* & 0 & E \\ 0 & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & C \\ 0 & 0 & E \\ C^* & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & E \\ 0 & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} A & B & 0 \\ 0 & - E ^2 & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & -CE^* & C \\ 0 & - E ^2 & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & 0 \\ 0 & - E ^2 & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	

TABLE III. Nine possible candidates for M_U or M_D with nonzero (1,1) element. Both hermitian and triangular forms are given in the hierarchical basis. Here $D' \equiv D + |E|^2$, and $A'' \equiv A + \frac{|CE|^2}{D} + |C|^2$.

Pattern	$(M_U^{11} \neq 0, M_D^{11} = 0)$	Relation	Viable
11	(M_9, M_7)	R7: $ V_{cb} \simeq \sqrt{\frac{m_c}{m_t}} \left 1 - \frac{V_{ub}}{V_{ts}^* K} \right ^{-\frac{1}{2}}$	Yes
12	(M_{10}, M_7)	R5: $\left \frac{V_{td}}{V_{ts}} \right \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
13	(M_{11}, M_3)	R4: $ V_{us} \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
14	(M_{12}, M_3)	R5: $\left \frac{V_{td}}{V_{ts}} \right \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
15	(M_{13}, M_4)	R4: $ V_{us} \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
16	(M_{14}, M_3)	R5: $\left \frac{V_{td}}{V_{ts}} \right \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
17	(M_{15}, M_3)	R8: $ V_{cb} \simeq \sqrt{\frac{m_c}{m_t}} \left 1 - \frac{V_{ub}}{V_{ts}^* K} \right ^{-1}$	Yes
18	(M_{16}, M_1)	R4: $ V_{us} \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
19	(M_{17}, M_1)	R4: $ V_{us} \simeq \sqrt{\frac{m_d}{m_s}}$	Yes
Pattern	$(M_U^{11} = 0, M_D^{11} \neq 0)$	Relation	Viable
20	(M_6, M_{10})	R9: $ V_{ub} \simeq \sqrt{\frac{m_u}{m_t}}$	Yes
21	(M_2, M_{12})	R10: $ V_{ub} \simeq \sqrt{\frac{m_u}{m_t}} \Gamma_{\pm}^{-\frac{1}{2}}$	#
22	(M_3, M_{12})	R6: $\left \frac{V_{ub}}{V_{cb}} \right \simeq \sqrt{\frac{m_u}{m_c}}$	No
23	(M_7, M_{10})	R6: $\left \frac{V_{ub}}{V_{cb}} \right \simeq \sqrt{\frac{m_u}{m_c}}$	No
24	(M_8, M_{10})	R6: $\left \frac{V_{ub}}{V_{cb}} \right \simeq \sqrt{\frac{m_u}{m_c}}$	No

TABLE IV. Possible hermitian pairs and their corresponding relations for $(M_U^{11} \neq 0, M_D^{11} = 0)$ and $(M_U^{11} = 0, M_D^{11} \neq 0)$. Here $\Gamma_{\pm}^{-1} \equiv 1 \pm V_{cb}^2 m_t / m_c \simeq 1 \pm 0.4$ at the scale m_Z . In writing relations R7 and R8, we have used for simplicity $V_{tb} \simeq V_{cs} \simeq V_{ud} \simeq 1$, and $K \equiv e^{i\delta} \sqrt{\frac{m_d}{m_s}} - \frac{V_{us}}{V_{cs}}$. The particular pair (pattern 21) that leads to predictions marginally acceptable by data is indicated by #. Note that no viable 4-texture-zero pair exists with $M_U^{11} \neq 0$ and $M_D^{11} \neq 0$.